The symmetric state of a rotating fluid differentially heated in the horizontal

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The motion of a fluid inside a rotating annulus of square cross-section, whose dimensions are small compared with the distance from the axis of rotation, is considered. The rigid side walls are held at different constant temperatures, and the fluid motions that occur are strongly influenced by Coriolis accelerations. A detailed study is made of the azimuthally independent state, a Hadley cell, in the limit of small thermal Rossby number. It is convenient to employ a boundary layer type analysis, essentially with respect to the Taylor number, and all the imposed boundary conditions are rigorously satisfied.

An entirely geostrophic thermal wind is found to obtain over the main body of the fluid. The circulation in the plane of the annular cross-section is entirely confined within narrow boundary layers and consists of a superposition of three cellular motions: a cell occupying the cross-section and two additional cells confined to the side-wall boundary layers. These motions are intimately related to the rotational constraint. The temperature distribution and its relation to the conductive and convective processes are determined.

1. The Hadley cell

1.1. *Introduction*

When a fluid is differentially heated in a horizontal direction, motions immediately occur, even if the imposed horizontal temperature difference is infinitesimal, since there is no pressure distribution capable of balancing the thermally induced gravitational body force. The simplest steady state attainable for a fluid of finite vertical extent with heating independent of one horizontal co-ordinate is that of a single cellular motion. The fluid ascends in the region of maximum heating, moves horizontally towards the region of maximum cooling, descends, and returns to its place of origin.

When the fluid is observed in a co-ordinate frame which is rotating relative to an inertial system, Coriolis acceleration produces a component of horizontal velocity perpendicular to the direction of the imposed thermal gradient. This component of velocity may be called the thermal wind. If rotation and heating rates are such that the Coriolis acceleration and the thermal body force are of the same order of magnitude, the thermal wind will be the largest component of velocity over the main body of the fluid.

The simple cellular motion of a horizontally heated rotating fluid bears the

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name of George Hadley (1735), who first made the above remarks in a discussion of the general circulation of the atmosphere.

1.2. *Hadley's original work*

The large-scale atmospheric motions have long been known to be driven by the latitudinal variation of the heating of the earth by the sun. However, the predominant feature of the resulting circulation is its longitudinal component. Hadley first related this phenomenon to the constraint of the earth's rotation in the paper referred to above, entitled 'Concerning the Cause of the General Trade-Winds'. In his words,

> 'I Think the Causes of the General Trade-Winds have not been fully explained by any of those who have wrote on that Subject, for want of more particularly and distinctly considering the Share the diurnal Motion of the Earth has in the Production of them....'†

Hadley considered the atmosphere to be thermally driven away from the state of solid rotation:

> 'For, let us suppose the Air in every Part to keep an equal Pace with the Earth in its diurnal Motion...then by the Action of the Sun on the Parts about the Equator, and the Rarefaction of the Air proceding therefrom, let the Air be drawn down thither from the N. and S. Parts.'‡

He then showed that easterlies would occur near the equator because

> '...the Air, as it moves from the Tropicks toward the Equator, having a less velocity than the Parts of the Earth it arrives at, will have a relative Motion contrary to that of the diurnal Motion of the Earth in these Parts...'§

Westerlies at mid-latitudes were similarly explained.

Hadley then made quantitative calculations of the magnitude of the winds assuming the air to keep the longitudinal velocity of the earth at the point of the airs origin. Finding the values thus calculated to be too large, he called upon surface friction:

> '...before the Air from the Tropicks can arrive at the Equator, it must have gained some Motion Eastward from the Surface of the Earth or Sea, whereby its relative Motion will be diminished, and in several successive Circulations, may be supposed to be reduced to the Strength it is found to be of.'‖

Hadley concluded

> 'That without the Assistance of the diurnal Motion of the Earth, Navigation, especially Easterly and Westerly, would be very tedious, and to make the whole Circuit of the Earth perhaps impracticable.'††

1.3. *Modelling in geophysical fluid dynamics*

The general circulation of the real atmosphere is of course much more complicated than a simple Hadley cell. Furthermore, theoretical studies are hampered not only because of the complexity of the governing mathematics, but also because meaningful quantitative data on real geophysical flows are very difficult to

† Hadley (1735), p. 58. ‡ *Ibid.* p. 60. § *Idem.* ‖ *Ibid.* p. 61. †† *Ibid.* p. 62.

obtain, especially on the large scale. Detailed theory could give information not obtainable observationally, but such theory is not useful unless it has first undergone a careful and detailed comparison with nature in some instances.

Recently, theoretical geophysical fluid dynamics has been stimulated by the initiation of controlled laboratory experiments in rotating tanks (Long 1953). These experiments, of interest in themselves, are also capable in some sense of modelling the larger scale, but the precise validity of the modelling remains an open question of great interest.

A class of experiments has been performed with a rotating fluid differentially heated in the horizontal. The initial motivation for these experiments lay in the possibility of their relation to the origin of the earth's magnetic field by convection in the liquid core (Hide 1958); and the meteorological interest aroused by early experimental results has led to extensive further work (Fultz 1956). The results are certainly a fascinating example of the physics of natural convection in the presence of rotation.

The annular space between two coaxial cylinders mounted on a rotating platform is filled with water. A horizontal temperature gradient is imposed on the boundaries. Three distinct types of steady states appear. First, a Hadley cell, or symmetric régime, in which the velocity and temperature fields are independent of the azimuthal angle about the axis of the cylinders is found. Secondly, a wave régime is found, i.e. a laminar flow in which the streamlines viewed from the top form a wave pattern, and various wave numbers appear. Thirdly, a turbulent régime is found with various numbers of well-defined eddies.

The régime and wave number selected by the fluid depends upon the imposed horizontal temperature difference and the rotation rate of the apparatus. For a given moderately strong rotation rate ($\Omega > 0{\cdot}1\,\mathrm{sec}^{-1}$), the symmetric régime occurs both for sufficiently small ($\Delta T < 0{\cdot}3\,^{\circ}\mathrm{C}$), and sufficiently large temperature differences. (These will be designated the lower and upper symmetric régimes.) At intermediate temperature differences the wave régime occurs. The transitions from régime to régime and from one wave number to another occur at definite critical values of the parameters. For large enough rotation rates, where the transition to the upper symmetric régime occurs at large temperature differences, the turbulent régime occurs between two wave régimes. A diagram of the various régimes is presented in figure 1. Since a full range of experiments has not as yet been performed on a given fluid in a given annular geometry, figure 1 is qualitatively inferred from several experiments.

To date, experiments, especially in the symmetric régime, have been mostly qualitative with some quantitative results for critical values of parameters governing the instabilities. Some ambiguity exists in the results because of lack of control of the boundary conditions, e.g. surface evaporation, wind stress, and uneven heating.

Meteorological interest in these experiments lies in the apparent relation of the wave and eddy régimes to the general circulation patterns in the atmosphere. However, to gain understanding of these complicated régimes by existing theoretical techniques, it is necessary first to have a complete theory of the symmetric state. Then by study of the stability of the symmetric state, the criterion for the initial onset of the wave régime can be found. Finally, it would be possible to attempt to determine the conditions which govern the preference of one particular wave number over another, a highly non-linear problem.

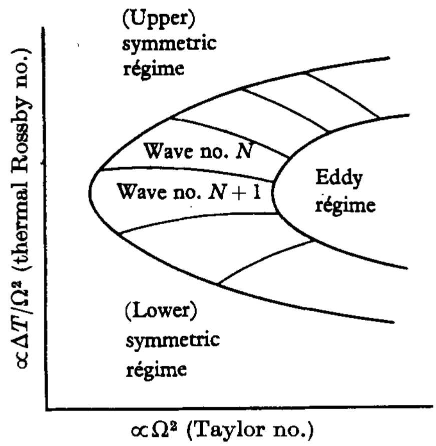

FIGURE 1. Qualitative picture of the various régimes.

Each step in this theoretical approach depends upon the preceding one and becomes succeedingly more difficult. Thus the theory of the Hadley cell, of interest *per se*, also provides the necessary firm foundation for a study of the more complicated states of a horizontally heated rotating fluid.

1.4. *The present problem*

In formulating a theoretical approach to a general class of phenomena, it is of course invaluable to study the simplest physical situation possible. We want the important aspects of the physics to be retained within the simplest framework in which their full implications can be realized, and also the situation considered must be simultaneously capable of precise experimental study and quantitative theoretical analysis.

In order for a fluid to exhibit a Hadley cell and the subsequent instabilities, it is important only that the fluid be heated from the sides and experience Coriolis accelerations. Therefore, it is of interest to consider a fluid confined to a rotating annular region of square cross-section, the dimension of which is small compared to the distance from the centre of rotation so that curvature and centrifugal effects will be minimized. The side walls are to be held at different constant temperatures, and the top and bottom walls are to be thermally insulating. Four rigid walls allow the elimination of extraneous boundary effects. In addition, the symmetry simplifies the physical situation and thereby reduces the mathematical complexity of its description. This model will be formalized in the next section.

Previous theoretical studies (e.g. Davies 1956; Kuo 1956) relevant to the general experimental situation described above differ from that indicated here. The situations discussed have been more complicated. Moreover, because these studies were primarily concerned with the stability problem, simplifying mathematical assumptions were made about the symmetric régime, and the assumptions used failed to bring boundary condition information into the final description. Aspects of the symmetric régime, such as internal temperature gradients, which are in fact determined by the boundary conditions, have been treated as free parameters in the stability problem.

2. The mathematical model

2.1. *Definitions and assumptions*

Consider a fluid contained between two coaxial cylinders and two parallel planes, the distance between the cylinders being equal to the distance between the planes. Let the container rotate with respect to an inertial system. The rotation vector, anti-parallel to gravity, coincides with the axis of the cylinders, both being normal to the planes (see figure 2).

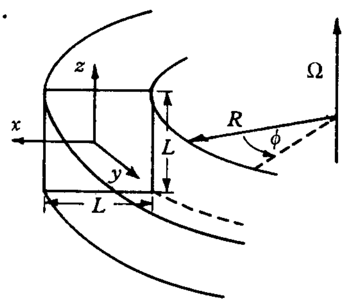

FIGURE 2. The co-ordinate system.

The fluid is thermally driven away from the state of solid rotation by an imposed horizontal temperature gradient, i.e. the inner and outer cylinders are held at different constant temperatures. The horizontal surfaces are thermally insulating.

We use the following nomenclature:

$\mathbf{r}$	Co-ordinate vector with components (x, y, z). With respect to the cylinders, x is radial, y is azimuthal, z is vertical. x and y are also referred to as meridional and zonal
$\mathbf{v}$	Velocity with components (u, v, w)
p	Deviation from hydrostatic pressure
T	Temperature. N.B. The above symbols primed have dimensions, unprimed they are dimensionless
Ψ	Stream function for the velocity components in the x-z plane. N.B. v, T, Ψ are also used with superscripts as coefficients in perturbation expansions
$(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$	Unit vectors in the (x, y, z) directions
g	Acceleration due to gravity
α	Coefficient of thermal expansion
ρ	Fluid density
ν	Kinematic viscosity
κ	Thermometric conductivity
Ω	Angular velocity of solid rotation
ΔT	Total horizontal temperature difference
R	Radius of inner cylinder
L	Side of square cross-section of annulus

The explicit assumptions defining the mathematical model applied to the fluid system are:

(*a*) A linear dependence of density on temperature alone as the equation of state:

$$\rho = \rho_0[1-\alpha(T'-T_0)]. \tag{2.1}$$

(*b*) The coefficient of thermal expansion, α, of the fluid is taken to be zero except when coupled with the gravitational constant. This allows for the thermally induced gravitational force, but neglects all other effects of density variation. In particular the assumption is made of local incompressibility,

$$\nabla'.\mathbf{v}' = 0. \tag{2.2}$$

Assumptions (*a*) and (*b*) are frequently used in similar problems and are known as the Boussinesq Approximation.

(*c*) Viscosity and thermal conductivity are constant.

(*d*) Frictional dissipation is neglected in the energy equation, which becomes, using also (*b*) and (*c*),

$$\kappa\nabla'^2T' - \mathbf{v}'.\nabla'T' = 0. \tag{2.3}$$

(*e*) The distance between the bounding cylinders is small compared to the radius of the inner cylinder, i.e. $L/R \ll 1$. This serves to eliminate curvature effects in the equations and implies a constant centrifugal term.

(*f*) The centrifugal acceleration is much smaller than the gravitational acceleration, i.e. $\Omega^2R/g \ll 1$. This is consistent with assumption (*b*) above.

Under these assumptions, the Navier–Stokes equations describing the symmetric ($\partial/\partial y' \equiv 0$) steady ($\partial/\partial t \equiv 0$) state take the form

$$-\nu\nabla'^2\mathbf{v}' + \mathbf{v}'.\nabla'\mathbf{v}' + 2\mathbf{\Omega}\times\mathbf{v}' - \alpha g T'\mathbf{k} + \rho_0^{-1}\nabla' p' = 0. \tag{2.4}$$

Here p' is the difference between the actual pressure and a pressure which balances the constant terms $\rho_0(g\hat{\mathbf{k}} - \Omega^2 R\hat{\mathbf{i}})$. The symbolic operators, of course, are only two-dimensional.

Equations (2.2), (2.3) and (2.4) provide five equations for the three components of velocity, the temperature and the pressure. Specifying the necessary boundary conditions completes the exposition of the formal mathematical problem. On the four rigid walls, all velocity components must vanish, i.e. $\mathbf{v}' = 0$ at $x' = \pm\frac{1}{2}L$, $z' = \pm\frac{1}{2}L$. The condition of constant temperature on the vertical walls is taken as $T' = \pm\frac{1}{2}(\Delta T)$ at $x' = \pm\frac{1}{2}L$; and no transfer of heat across the horizontal walls implies $\partial T'/\partial z' = 0$ at $z' = \pm\frac{1}{2}L$.

2.2. *The method of approximation*

Although in § 1 the simplest physical problem for the symmetric circulation of a horizontally heated rotating fluid has been formally posed, it is still too difficult mathematically for direct solution. The equations are coupled, high order, non-linear partial differential equations. In order to proceed, some approximate procedure must be adopted.

Since any approximation will consist of arguments concerning relative orders of magnitude, it will be convenient to introduce non-dimensional variables and parameters at this point. Let $\mathbf{r}' = L\mathbf{r}$, $T' = (\Delta T)\,T$, $\mathbf{v}' = c\mathbf{v}$, and $p' = c^2\rho_0 p$, where c is an as yet undetermined velocity. When these variables are substituted into (2.4), it can be seen that the choice of $c = \alpha g(\Delta T)(2\Omega)^{-1}$ will transform the terms $2\mathbf{\Omega}\times\mathbf{v}'-\alpha g T'\hat{\mathbf{k}}$ into $\hat{\mathbf{k}}\times\mathbf{v}-T\hat{\mathbf{k}}$, i.e. into terms which are (non-dimensionally) of order unity. This is appropriate because $T\hat{\mathbf{k}}$ represents the driving force, and also because interest lies in the case where the constraint of rotation, $\hat{\mathbf{k}}\times\mathbf{v}$, is of the same order of magnitude.

It is also convenient now to make use of the azimuthal symmetry in the continuity equation (2.2), which becomes $\partial u/\partial x+\partial w/\partial z = 0$. Thus it is possible to introduce a stream function for the velocity components in the x-z plane defined by $u \equiv \Psi_z$, $w \equiv -\Psi_x$, subscripts denoting partial differentiation. The boundary conditions on the velocity now require that Ψ and its normal derivative be zero on all boundaries.

Introducing the non-dimensional variables and the stream function, and subtracting after cross-differentiating the first and third components of (2.4) to eliminate the pressure, we obtain the remaining equations:

$$-\epsilon\nabla^4\Psi+\beta[\Psi_z\Psi_{xzz}+\Psi_z\Psi_{xxx}-\Psi_x\Psi_{zzz}-\Psi_x\Psi_{xxz}]-v_z+T_x = 0, \tag{2.5}$$

$$-\epsilon\nabla^2 v+\beta[\Psi_z v_x-\Psi_x v_z]+\Psi_z = 0, \tag{2.6}$$

$$-\epsilon\nabla^2 T+\beta\sigma[\Psi_z T_x-\Psi_x T_z] = 0, \tag{2.7}$$

where $\epsilon = \nu(2\Omega)^{-1}L^{-2}$ is the reciprocal square-root of a Taylor number, $\beta = \alpha g(\Delta T)(2\Omega)^{-2}L^{-1}$ is a thermal Rossby number, and $\sigma = \nu\kappa^{-1}$ is the Prandtl number.

The lower symmetric régime is characterized by small total temperature difference and, in the range of interest where instabilities are known to occur in related experiments, by moderately strong rotation rates; thus this régime is associated with small values of β and ϵ. We now assume that $\beta < 1$, $\epsilon < 1$, and $\sigma = O(1)$, but since these properties of the parameters are used to develop approximate solutions, more explicit limitations on their magnitudes will be found later.

Ordinary perturbation expansions in β of all the variables can now be made, i.e.

$$\Psi = \sum_{n=0}^{\infty}\beta^n\Psi^{(n)},\quad v = \sum_{n=0}^{\infty}\beta^n v^{(n)},\quad T = \sum_{n=0}^{\infty}\beta^n T^{(n)}.$$

A single superscript (n) denotes the nth order β-expansion coefficient for the variable on which it appears. The expansions are inserted into equations (2.5), (2.6) and (2.7), and terms of order β^n are collected.

The terms of zero order give:

$$-\epsilon\nabla^4\Psi^{(0)}-v_z^{(0)}+T_x^{(0)} = 0, \tag{2.8}$$

$$-\epsilon\nabla^2 v^{(0)}+\Psi_z^{(0)} = 0, \tag{2.9}$$

$$\nabla^2 T^{(0)} = 0. \tag{2.10}$$

The terms of first order give:

$$\epsilon\nabla^4\Psi^{(1)}-v_z^{(1)}+T_x^{(1)}+[\Psi_z^{(0)}\Psi_{xzz}^{(0)}+\Psi_z^{(0)}\Psi_{xxx}^{(0)}-\Psi_x^{(0)}\Psi_{zzz}^{(0)}-\Psi_x^{(0)}\Psi_{xxz}^{(0)}] = 0, \tag{2.11}$$

$$-\epsilon\nabla^2 v^{(1)}+\Psi_z^{(1)}+[\Psi_z^{(0)}v_x^{(0)}-\Psi_x^{(0)}v_z^{(0)}] = 0, \tag{2.12}$$

$$-\epsilon\nabla^2 T^{(1)}+\sigma[\Psi_z^{(0)}T_x^{(0)}-\Psi_x^{(0)}T_z^{(0)}] = 0, \tag{2.13}$$

and so on. Each $\Psi^{(n)}$ separately satisfies the boundary conditions on Ψ. $T^{(0)}$ satisfies the non-dimensional condition $T^{(0)} = \pm\frac{1}{2}$ at $x = \frac{1}{2}$, $T^{(n)} = 0$ at $x = \pm\frac{1}{2}$ for $n > 0$. At $z = \pm\frac{1}{2}$, $T_z^{(n)} = 0$ for all n.

From the form of these equations, the role of β in the heat equation, as a measure of the relative importance of conduction and convection in determining the temperature distribution, is made clear. Small β means that conduction effects are important. The zero-order temperature is in fact determined entirely by conduction, and thus is not coupled to the zero-order velocity. This temperature distribution then acts as the forcing term for the coupled zero-order velocity equations (2.8) and (2.9). Next, the first-order temperature is determined by conduction balancing advection of the zero-order temperature field by the zero-order velocities. Similarly, to any order in β, the temperature is first determined, by conduction, from a Poisson equation in which the inhomogeneity consists of advections of lower-order temperatures by lower-order velocities. The known temperature then appears as an inhomogeneity in the coupled velocity equations of the same order, which are also forced by lower-order momentum advections. The latter will be found to be relatively unimportant.

The mathematical difficulties have now been considerably lessened. All the equations have been linearized, and the heat equation has been reduced to a simple standard form. However, the coupled velocity equations are not simple, and it will be worthwhile to exploit the fact that ϵ is small in their solution. Since ϵ multiplies the highest order differentiated terms in each equation, a direct expansion in this parameter is not permissible. If this were done, the order of the equations would be lowered and the boundary conditions could not be satisfied. Singular perturbation theory or boundary-layer analysis must be used (see Carrier, 1953).†

The highest-order differentiated terms can contribute only when they are large enough to overcome the small factor ϵ. Extreme changes of the velocity functions and their derivatives are not expected over the main body of the fluid, but can occur near the boundaries, where viscous effects are important. Therefore solutions of approximate equations of low order will be tolerated in the interior of the fluid only. Near each boundary additional contributions to the solutions will appear; these additional contributions will satisfy approximate equations of higher order, and together with the interior solutions will satisfy all boundary conditions.

Since the zero-order velocities are expected to have a boundary-layer character, and these velocities force the first-order temperature, it is expected that first- and higher-order temperatures will also exhibit boundary-layer effects. It is pertinent to notice that in equation (2.13) the same small parameter ϵ also couples the highest-order differentiated terms in the temperature.

In the next section, solutions for the lower symmetric régime will be obtained by the double perturbation procedure outlined above.

† This reference should be consulted for any questions in standard boundary-layer techniques which are employed in the following analysis.

3. Analysis and results

3.1. *The zero order*

The zero-order temperature may be trivially obtained from equation (2.10) because of the particularly simple boundary conditions imposed. Thus

$$T^{(0)}(x, z) = x, \tag{3.1}$$

and $T^{(0)}_x = 1$ is to be inserted into (2.8).

Treating (2.8) and (2.9) as a boundary-layer problem, we must first determine the boundary-layer widths and the amplitudes as functions of ϵ. Since the equations are linear, the boundary-layer widths, or more precisely the characteristic lengths normal to the boundaries in which the highly differentiated terms contribute, can be determined independently of amplitude considerations. This is done most simply by considering the single equation satisfied by either $\Psi^{(0)}$ or $v^{(0)}$ alone, e.g.

$$\epsilon^2 \nabla^6 \Psi^{(0)} + \Psi^{(0)}_{zz} = 0, \tag{3.2}$$

which is obtained by cross-differentiation.

Scaled normal co-ordinates are defined near each of the boundaries by $\xi = \epsilon^a(x + \frac{1}{2})$ near $x = -\frac{1}{2}$ and $\zeta = \epsilon^b(z + \frac{1}{2})$ near $z = -\frac{1}{2}$. The exponents a and b will be determined by deriving appropriate approximate equations for the contributions near each wall from (3.2). Negative values are anticipated so that ξ and ζ will be stretched co-ordinates, i.e. a large change in ξ will be equivalent to a small change in x. Equation (3.2) is now rewritten in terms of ξ and z, and x and ζ as independent variables. The derivatives $\partial/\partial x$ and $\partial/\partial z$ are expressed as $\epsilon^{-a}(\partial/\partial \xi)$ and $\epsilon^{-b}(\partial/\partial \zeta)$ respectively. The assumption that $\Psi^{(0)}$ is smooth, i.e. the function itself and all its derivatives are bounded above by order unity, is made. The equations can then be ordered with respect to ϵ as a small quantity. To perform the ordering, the highest term of ∇^6 is required in each case to balance the rotational constraint, the last term in (3.2). This procedure yields $a = -\frac{1}{3}$, $b = -\frac{1}{2}$, and the approximate equations

$$\Psi^{(0)}_{\xi\xi\xi\xi\xi\xi} + \Psi^{(0)}_{zz} = 0, \quad \text{near } x = -\tfrac{1}{2}, \tag{3.3}$$

$$\Psi^{(0)}_{\zeta\zeta\zeta\zeta\zeta\zeta} + \Psi^{(0)}_{\zeta\zeta} = 0, \quad \text{near } z = -\tfrac{1}{2}. \tag{3.4}$$

The first neglected terms in (3.3) are $O(\epsilon^{\frac{2}{3}})$, and in (3.4) are $O(\epsilon)$.

The same equations in similarly defined stretched co-ordinates will pertain near $x = \frac{1}{2}$, $z = \frac{1}{2}$. It will be noticed that the side-wall boundary layers, of width $O(\epsilon^{\frac{1}{3}})$, are broader than the top and bottom boundary layers, of width $O(\epsilon^{\frac{1}{2}})$. The approach adopted here will, of course, break down in the corner regions where none of the approximations are valid. However, these regions are small and relatively unimportant. The vertical and horizontal boundary layers will be mutually consistent because they will uniquely satisfy, when added to the same interior solution, boundary conditions along their respective walls.

To determine the amplitudes of the contributions to the fields in the separate regions, it is necessary to return to the coupled equations which also contain the forcing inhomogeneity. It is convenient to transfer the inhomogeneity into the boundary conditions. This is readily done by extracting the particular solution with the substitution $v^{(0)} = z + v^{(0)}_h$. The boundary conditions change to $v^{(0)}_h = -z$

at $x = \pm\frac{1}{2}$; $v_h^{(0)} = \mp\frac{1}{2}$ at $z = \pm\frac{1}{2}$. The particular solution is recognized as the geostrophic thermal wind, well known in meteorology. To isolate the dependence of the amplitudes on ϵ, let

$$\Psi^{(0)} = A(\epsilon)\,\psi^{(0)}, \quad v_h^{(0)} = B(\epsilon)\,v^{(0)}, \quad \text{in the interior;}$$
$$_{(1)}\Psi^{(0)} = A_1(\epsilon)\,_{(1)}\psi^{(0)}, \quad _{(1)}v_h^{(0)} = B_1(\epsilon)\,_{(1)}v^{(0)}, \quad \text{near the vertical wall;}$$
$$_{(2)}\Psi^{(0)} = A_2(\epsilon)\,_{(2)}\psi^{(0)}, \quad _{(2)}v^{(0)} = B_2(\epsilon)\,_{(2)}v^{(0)}, \quad \text{near the horizontal wall.}$$

For convenience the boundary-layer region near the vertical wall will hereafter be referred to as region 1, and that near the horizontal wall as region 2. A lower left-hand subscript on a variable refers to the region. $_{(i)}\psi^{(0)}$ and $_{(i)}v^{(0)}$ are assumed to be independent of ϵ, and to be smooth in their arguments. The above expressions are now substituted into the approximate forms of (2.8) and (2.9), using the scaling information obtained above, yielding:

In the interior:
$$Bv_z^{(0)} = 0, \tag{3.5}$$
$$A\psi_z^{(0)} = 0. \tag{3.6}$$

In region 1:
$$\epsilon^{-\frac{1}{3}}A_1\,_{(1)}\psi^{(0)}_{\xi\xi\xi\xi} + B_1\,_{(1)}v_z^{(0)} = 0, \tag{3.7}$$
$$\epsilon^{\frac{1}{3}}B_1\,_{(1)}v^{(0)}_{\xi\xi} - A_1\,_{(1)}\psi_z^{(0)} = 0. \tag{3.8}$$

In region 2:
$$\epsilon^{-1}A_2\,_{(2)}\psi^{(0)}_{\zeta\zeta\zeta\zeta} + \epsilon^{-\frac{1}{2}}B_2\,_{(2)}v_\zeta^{(0)} = 0, \tag{3.9}$$
$$B_2\,_{(2)}v^{(0)}_{\zeta\zeta} - A_2\,\epsilon^{-\frac{1}{2}}\,_{(2)}\psi_\zeta^{(0)} = 0. \tag{3.10}$$

These homogeneous equations† yield relationships between the amplitudes of $\Psi^{(0)}$ and $v_h^{(0)}$, which are required to make all terms in a given equation consistently the same order in ϵ.

The interior equations contain no amplitude information. However, the boundary-layer equations (3.7) and (3.8) give, consistently,

$$A_1 = \epsilon^{\frac{1}{3}}B_1, \tag{3.11}$$

and, (3.9) and (3.10),
$$A_2 = \epsilon^{\frac{1}{2}}B_2. \tag{3.12}$$

The boundary conditions further require that at least part of $v_h^{(0)}$ be of order unity near each wall; thus
$$B_1 = 1 \quad (A_1 = \epsilon^{\frac{1}{3}}), \tag{3.13}$$
$$B_2 = 1 \quad (A_2 = \epsilon^{\frac{1}{2}}), \tag{3.14}$$

in the appropriate regions. If relationships (3.13) and (3.14) had happened to be identical, it would have been relatively simple to proceed from this point. Since they are not, further argument is necessary.

The only factor affecting amplitudes not contained in the above general statements is the requirement that the boundary conditions be satisfied by the boundary-layer contributions near each wall added to the same interior solution. In the formal expansion which is being evolved here, boundary conditions must of course be separately satisfied by all orders in ϵ. Thus amplitudes in one region strongly influence amplitudes in another region. In considering the effect of the

† The homogeneous equations are definitely correct for the contributions, of lowest order in ϵ, to $\psi^{(0)}$ and $v^{(0)}$ in each region. The equations for higher-order terms may be modified by the appearance of inhomogeneities in the form of the neglected parts of ∇^4 and ∇^2 operating on the lower-order fields.

application of both vertical and horizontal wall boundary conditions on the amplitude problem, certain particular properties of the solutions to the governing equations (3.5–3.10) are necessary and will now be derived. These equations are taken as independent of ϵ and solved under the auxiliary restrictions (3.11, 3.12).

The interior equations (3.5, 3.6) integrate immediately to

$$v^{(0)} = f(x), \tag{3.15}$$

$$\psi^{(0)} = g(x). \tag{3.16}$$

The boundary-layer equations in region 2, (3.9, 3.10),† can be treated as ordinary equations (with constant coefficients) in ζ with x-dependent integration constants. Rejecting solutions which do not approach zero asymptotically in ζ, and requiring the normal derivative of ${}_{(2)}\psi^{(0)}$ to vanish on the wall, we find the solutions to be

$${}_{(2)}\psi^{(0)} = G(x)\, e^{-\zeta/\sqrt{2}}(\sin \zeta/\sqrt{2} + \cos \zeta/\sqrt{2}), \tag{3.17}$$

and

$${}_{(2)}v^{(0)} = -\sqrt{2}\, G(x)\, e^{-\zeta/\sqrt{2}} \cos \zeta/\sqrt{2}. \tag{3.18}$$

Thus at $\zeta = 0$,

$${}_{(2)}\psi^{(0)} = G(x), \tag{3.19}$$

and

$${}_{(2)}v^{(0)} = -\sqrt{2}\, G(x). \tag{3.20}$$

Near the top boundary, let $\zeta^* = \epsilon^{-\frac{1}{2}}(z-\frac{1}{2})$; then the form of equations (3.9, 3.10) is reproduced in ζ^* under the transformation

$$\zeta \to -\zeta^*, \quad {}_{(2)}\psi^{(0)} \to {}_{(2)}\psi^{(0)}, \quad {}_{(2)}v^{(0)} \to -{}_{(2)}v^{(0)}.$$

Thus along the top wall,

$${}_{(2)}\psi^{(0)} = G^*(x), \tag{3.21}$$

and

$${}_{(2)}v^{(0)} = \sqrt{2}\, G^*(x). \tag{3.22}$$

Those properties of the solutions which have been obtained now enable the amplitude argument to be concluded. The boundary conditions require that values of $v_h^{(0)}$ of order unity exist in both regions 1 and 2, i.e. the existence of $B_1 = 1$ and $B_2 = 1$. In turn, the amplitude consistency relations (3.15, 3.16) then require $A_1 = \epsilon^{\frac{1}{3}}$ and $A_2 = \epsilon^{\frac{1}{2}}$ in their respective regions. A boundary-layer contribution to $\Psi^{(0)}$ which will add to an interior contribution to give a zero stream function on the walls forces the amplitude of the interior stream function, i.e. requires definite values of A. Since the amplitudes of $\Psi^{(0)}$ required in the different boundary layers by the above arguments are not of the same order in ϵ, they cannot simply add together with the same interior solution.

To proceed, it would seem necessary to require contributions to the interior $\Psi^{(0)}$ of both amplitudes, i.e. to have in the interior $\Psi^{(0)} = \epsilon^{\frac{1}{3}}\psi^{(0,\frac{1}{3})} + \epsilon^{\frac{1}{2}}\psi^{(0,\frac{1}{2})} + \ldots$. (Hereafter when two superscripts are used the first will refer to the order in β and the second to the order in ϵ. A single superscript will continue to refer to the order in β.) However, the appearance of an amplitude of $\Psi^{(0)}$ in the interior further requires the same amplitude in the boundary layer other than whence it originated. E.g. $A_1 = \epsilon^{\frac{1}{3}}$ implies $A = \epsilon^{\frac{1}{3}}$ which in turn implies $A_2 = \epsilon^{\frac{1}{3}}$, etc.

Invoking again the amplitude consistency relationships (3.11, 3.12), we see that $A_2 = \epsilon^{\frac{1}{3}}$ requires a contribution to ${}_{(2)}v_h^{(0)}$ of $B_2 = \epsilon^{-\frac{1}{6}}$. This in turn would carry a

† These equations are controlled by the rotational constraint in a manner first noticed by Ekman (1905) while considering the effect of the wind stress on the ocean.

$v_h^{(0)}$ of order $\epsilon^{-\frac{1}{3}}$ through the interior and into the side-wall boundary layer. Thus the fact that (3.11) and (3.12) differ would generate, indefinitely, further amplitudes of both $\Psi^{(0)}$ and $v_h^{(0)}$ in all three regions.

The possibility of simplification arises only if one of the originally required contributions to $\Psi^{(0)}$ satisfies the boundary conditions by itself at the wall of the boundary-layer region in which it necessarily exists. Note that this is physically interpreted as allowing for the possibility of a boundary-layer counter-current. If either ${}_{(1)}\psi^{(0,\frac{1}{3})} = 0$ at $\xi = 0$ or ${}_{(2)}\psi^{(0,\frac{1}{2})} = 0$ at $\zeta = 0$, a contribution of the same amplitude from the interior, etc., is not required. Finally, the properties of ${}_{(2)}\psi^{(0,\frac{1}{2})}$ revealed in (3.17, 3.19) show that ${}_{(2)}\psi^{(0,\frac{1}{2})}$ cannot be zero at $\zeta = 0$. Such a requirement would force ${}_{(2)}\psi^{(0,\frac{1}{2})}$ to be identically zero everywhere in region 2, which is impossible.

However, it will be seen that ${}_{(1)}\psi^{(0,\frac{1}{3})}$ can meet this requirement in region 1. Therefore, the following problem is formulated as being the simplest consistent with the full amplitude and scaling requirements. The unique and unambiguous results obtained below indicate that it is the correct boundary-layer problem associated with equations (2.8, 2.9), and therefore provides formal approximate solutions to these equations.

In the interior and in region 2, near the horizontal wall, the stream function is $O(\epsilon^{\frac{1}{2}})$ and the zonal velocity is $O(1)$, $A = A_2 = \epsilon^{\frac{1}{2}}$, and $B = B_2 = 1$. The boundary conditions at $z = -\frac{1}{2}$ are applied to $\Psi^{(0)} = \epsilon^{\frac{1}{2}}(\psi^{(0,\frac{1}{2})} + {}_{(2)}\psi^{(0,\frac{1}{2})})$ and $v_h^{(0)} = v^{(0,0)} + {}_{(2)}v^{(0,0)}$, and similarly at $z = +\frac{1}{2}$. Near the vertical wall in region 1, two amplitudes of the stream function and two amplitudes of the zonal velocity are required, $A_1 = \epsilon^{\frac{1}{3}}$ and $\epsilon^{\frac{1}{2}}$, and $B_1 = 1$ and $\epsilon^{\frac{1}{6}}$, i.e. ${}_{(1)}\psi^{(0)} = \epsilon^{\frac{1}{3}}\,{}_{(1)}\psi^{(0,\frac{1}{3})} + \epsilon^{\frac{1}{2}}\,{}_{(1)}\psi^{(0,\frac{1}{2})}$, ${}_{(1)}v_h^{(0)} = {}_{(1)}v^{(0,0)} + \epsilon^{\frac{1}{6}}\,{}_{(1)}v^{(0,\frac{1}{6})}$. Each set ${}_{(1)}\psi^{(0,\frac{1}{3})}$, ${}_{(1)}v^{(0,0)}$ and ${}_{(1)}\psi^{(0,\frac{1}{2})}$, ${}_{(1)}v^{(0,\frac{1}{6})}$ separately satisfies equations (3.7, 3.8). The boundary conditions at $x = -\frac{1}{2}$ are $\psi^{(0,\frac{1}{2})} + {}_{(1)}\psi^{(0,\frac{1}{2})} = 0$, ${}_{(1)}\psi^{(0,\frac{1}{3})} = 0$, $v^{(0,0)} + {}_{(1)}v^{(0,0)} = -z$, ${}_{(1)}v^{(0,\frac{1}{6})} = 0$. These boundary conditions are symmetrical in x, and introducing near $x = +\frac{1}{2}$ the variable $\xi^* = \epsilon^{-\frac{1}{3}}(x-\frac{1}{2})$, the form of (3.7, 3.8) is reproduced in ξ^* under the transformation $\xi \to -\xi^*$. Thus solutions near $x = \frac{1}{2}$ are immediately available from those at $x = -\frac{1}{2}$.

Applying the boundary conditions at the top and bottom walls to the solution (3.15, 3.16, 3.19–3.22), we obtain

$$G(x) + g(x) = 0,$$
$$G^*(x) + g(x) = 0,$$
$$-\sqrt{2}\,G(x) + f(x) = -\tfrac{1}{2},$$
$$\sqrt{2}\,G^*(x) + f(x) = +\tfrac{1}{2},$$

which have the solutions

$$g(x) = -G(x) = -G^*(x) = \frac{1}{2\sqrt{2}}, \tag{3.23}$$

and

$$f(x) = 0. \tag{3.24}$$

Thus the total zero order interior velocity is purely the geostrophic thermal wind. The velocity in the x-z plane, the vertical and meridional flow, is identically zero to this order in ϵ and β.

It remains to find the solutions near the side walls. The boundary conditions are now further specified in terms of equations (3.23, 3.24) as ${}_{(1)}\psi^{(0,\frac{1}{2})} = -1/(2\sqrt{2})$,

${}_{(1)}v^{(0,0)} = -z$. With the conditions imposed, the solutions are most readily obtained in terms of Fourier series in the vertical co-ordinate, the Fourier coefficients being explicit functions of the boundary-layer co-ordinate. Separated solutions of the constant coefficient equations (3.7, 3.8) exist of the form $\exp\{(k^2)^{\frac{1}{6}}\xi\}\begin{smallmatrix}\sin\\ \cos\end{smallmatrix} kz$, and the boundary conditions can be applied in terms of well-known square wave and sawtooth Fourier series. The solutions are

$$
\begin{aligned}
{}_{(1)}\psi^{(0,\frac{1}{3})} = \frac{1}{2\sqrt{2}}\left(\frac{16\sqrt{2}}{3\sqrt{3}\,\pi^{\frac{7}{3}}}\right)\sum_{n=0}^{\infty}(2n+1)^{-\frac{7}{3}}\Big\{\sqrt{3}\exp\{-[(2n+1)\pi]^{\frac{1}{3}}\xi\} \\
+\exp\{-[(2n+1)\pi]^{\frac{1}{3}}\}\frac{\xi}{2}\left(-\sqrt{3}\cos[(2n+1)\pi]^{\frac{1}{3}}\frac{\sqrt{3}}{2}\xi+\sin[(2n+1)\pi]^{\frac{1}{3}}\frac{\sqrt{3}}{2}\xi\right)\Big\} \\
\times(-)^n\cos(2n+1)\pi z,
\end{aligned}
\tag{3.25}
$$

$$
\begin{aligned}
{}_{(1)}v^{(0,0)} = \left(\frac{8}{3\sqrt{3}\,\pi^{2}}\right)\sum_{n=0}^{\infty}(2n+1)^{-2}\Big\{\sqrt{3}\exp\{-[(2n+1)\pi]^{\frac{1}{3}}\xi\} \\
+\tfrac{1}{2}\exp\{-[(2n+1)\pi]^{\frac{1}{3}}\}\frac{\xi}{2}\left(\sqrt{3}\cos[(2n+1)\pi]^{\frac{1}{3}}\frac{\sqrt{3}}{2}\xi+\sin[(2n+1)\pi]^{\frac{1}{3}}\frac{\sqrt{3}}{2}\xi\right)\Big\} \\
\times(-)^{n+1}\sin(2n+1)\pi z,
\end{aligned}
\tag{3.26}
$$

and

$$
\begin{aligned}
{}_{(1)}\psi^{(0,\frac{1}{2})} = \frac{1}{2\sqrt{2}}\left(\frac{-4}{\sqrt{3}\,\pi}\right)\sum_{n=0}^{\infty}(2n+1)^{-1}\exp\{-[(2n+1)\pi]^{\frac{1}{3}}\}\frac{\xi}{2} \\
\times\left(\sqrt{3}\cos[(2n+1)\pi]^{\frac{1}{3}}\frac{\sqrt{3}}{2}\xi+\sin[(2n+1)\pi]^{\frac{1}{3}}\frac{\sqrt{3}}{2}\xi\right)(-)^n\cos(2n+1)\pi z,
\end{aligned}
\tag{3.27}
$$

$$
\begin{aligned}
{}_{(1)}v^{(0,\frac{1}{6})} = \left(\frac{\sqrt{2}}{\sqrt{3}\,\pi^{\frac{2}{3}}}\right)\sum_{n=0}^{\infty}(2n+1)^{-\frac{2}{3}} \\
\times\exp\{-[(2n+1)\pi]^{\frac{1}{3}}\}\frac{\xi}{2}\left(\sin[(2n+1)\pi]^{\frac{1}{3}}\frac{\sqrt{3}}{2}\xi\right)(-)^{n+1}\sin(2n+1)\pi z.
\end{aligned}
\tag{3.28}
$$

These results are plotted and discussed more fully in § 3.4 below. ${}_{(1)}\psi^{(0,\frac{1}{2})}$ represents a flow up the hot wall and down the cold wall, the streamlines of the flow are bowed out from the wall, the maximum bowing occurring at the mid-point of the wall. All the mass transport in the side boundary layers is associated with ${}_{(1)}\psi^{(0,\frac{1}{2})}$. ${}_{(1)}\psi^{(0,\frac{1}{3})}$ represents a narrow circulating cell of velocity confined to the vertical wall boundary layers.† This is an interesting unanticipated result directly related to the rotational constraint.

3.2. *The first order*

The first-order temperature calculation is of particular interest because it contains the first modifications of the temperature from that governed purely by conduction. In physical situations where the perturbation calculations evolved here are expected to be useful, the observable temperature will be the temperature through first order in β. Thus the vertical temperature gradient of importance to the anticipated stability problem will be a result of this calculation.

† *Note added in proof.* From a consideration of the asymptotic form of the exact solution to equation (3.2), a single contribution to this cell of width $\epsilon^{\frac{1}{4}}$ is found. Thus ${}_{(1)}\psi^{(0,\frac{1}{3})}$ is partly spurious. This failure of boundary-layer analysis is discussed more fully elsewhere (Carrier & Robinson, to be published). It corresponds solely to a modification of the non-mass-transporting cell confined to the side-wall boundary layer. No other results are affected. For an example of the appearance of an $\epsilon^{\frac{1}{4}}$ contribution in a similar problem, see Stewartson (1957).

The zero-order convection terms in (2.13) forcing the first-order temperature can be evaluated from the results of the preceding section. Since $T_z^{(0)} = 0$ and $T_x^{(0)} = 1$ everywhere, the last term of (2.13) is identically zero and $T^{(1)}$ is determined by the equation
$$-\epsilon\nabla^2(\sigma^{-1}T^{(1)}) + \Psi_z^{(0)} = 0. \tag{3.29}$$

On comparison with equation (2.9) for the zero-order zonal velocity, $v^{(0)}$, (3.29) is seen to be identical with $\sigma^{-1}T^{(1)}$ replacing $v^{(0)}$. Therefore a particular solution of (3.29) is $T_p^{(1)} = \sigma v^{(0)}$. As anticipated, this contribution to the temperature forms strong boundary layers. With respect to ϵ, $T_p^{(1)}$ is $O(1)$ everywhere with an additional contribution $O(\epsilon^{\frac{1}{2}})$ near the constant temperature side walls, region 1.

To preserve the physical conditions on the total temperature, the first-order temperature at the walls must obey the conditions $T^{(1)} = 0$ at $x = \pm\frac{1}{2}$; $\partial T^{(1)}/\partial z = 0$ at $z = \pm\frac{1}{2}$. Since $v^{(0)}$ is zero on all walls, the first condition is obeyed by $T_p^{(1)}$, but the second condition is not. From (3.18) and (3.23),

$$\frac{\partial}{\partial z}T_p^{(1)}|_{z=-\frac{1}{2}} = \epsilon^{-\frac{1}{2}}\frac{\sigma}{z}\frac{\partial}{\partial\zeta}(\exp\{-\zeta/\sqrt{2}\}\cos\zeta/\sqrt{2}|_{\zeta=0} + O(1) = -\frac{\epsilon^{-\frac{1}{2}}\sigma}{2\sqrt{2}} \tag{3.30}$$

to the lowest order in ϵ. $T_p^{(1)}$ is antisymmetric in z; therefore

$$\frac{\partial}{\partial z}T_p^{(1)}|_{z=+\frac{1}{2}} = -\frac{\epsilon^{-\frac{1}{2}}\sigma}{2\sqrt{2}}$$

also. Thus homogeneous solutions to (3.29) must contribute to $T^{(1)}$, and must satisfy the boundary conditions

$$T_h^{(1)} = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}; \quad \frac{\partial}{\partial z}T_h^{(1)} = +\frac{\epsilon^{-\frac{1}{2}}\sigma}{2\sqrt{2}} \quad \text{at} \quad z = \pm\tfrac{1}{2}.$$

Since $T_h^{(1)}$ satisfies the homogeneous Laplace equation, it cannot form boundary layers, i.e. have extremely sharp gradients near the walls. Therefore the condition in ϵ on the gradient of $T_h^{(1)}$ on the top and bottom walls must be taken as a condition on the amplitude of $T_h^{(1)}$ everywhere. Thus $T_h^{(1)}(x, z) = \epsilon^{-\frac{1}{2}}T^{(1,-\frac{1}{2})}(x, z)$, where $\nabla^2 T^{(1,-\frac{1}{2})} = 0$. Since the temperature $T^{(1,-\frac{1}{2})}$ is $\epsilon^{-\frac{1}{2}}$ larger than the directly convected temperature $T_p^{(1)} = T^{(1,0)}$, then $T^{(1,-\frac{1}{2})}$ will be the observed temperature. Thus in this case, the vertical temperature distribution is primarily set directly by conduction, but is, nevertheless, implicitly controlled by convective processes. Conduction acts to maintain the physical boundary conditions disturbed by convection.

Anticipating the first-order velocity calculation to be forced by this temperature distribution, it is most convenient to obtain $T^{(1,-\frac{1}{2})}$ as a Fourier series in the vertical co-ordinate. This is readily done by applying a finite Fourier transform. The result is

$$T^{(1,-\frac{1}{2})} = \frac{\sigma}{2\sqrt{2}}\left\{\left(\frac{4}{\pi^2}\right)\sum_{n=0}^{\infty}\left[\frac{(-)^{n+1}\cosh(2n+1)\pi x\sin(2n+1)\pi z}{(2n+1)^2\cosh(2n+1)\frac{1}{2}\pi}\right] + z\right\}. \tag{3.31}$$

The isotherms of this temperature distribution plotted in §3.4 are seen to be almost horizontal over the main body of the fluid. Thus the isotherms of the total temperature, to first order, will be tilted away from the vertical zero-order conduction isotherms. The shape of the tilted isotherms is given by the purely numerical function $\sigma^{-1}T^{(1,-\frac{1}{2})}$. The magnitude of the deviation from the pure conduction temperature is conveniently expressed by the multiplicative factor $\beta\epsilon^{-\frac{1}{2}}\sigma$.

To obtain the first-order velocities, equations (2.12, 2.13) are employed. A boundary-layer analysis entirely similar to that used in solving the zero-order equations (2.8, 2.9) can be used, even in the presence of the more complicated inhomogeneous terms now present. The inhomogeneous forcing terms consist of zero-order inertial terms and the first-order horizontal temperature gradient. From the zero-order solutions, the inertial terms in (2.11, 2.12) are found to be identically zero except near the side walls. In region 1, the inertial terms in (2.11) are at most $O(\epsilon^{-\frac{1}{3}})$, and in (2.12) at most $O(1)$. The contributions to $\Psi^{(1)}$ and $v^{(1)}$ forced by these terms will be of lower order in ϵ than the contributions forced by $T_x^{(1)}$ considered below, and may therefore be formally neglected.

Since the equations are linear, the contributions to the velocities from $T^{(1,-\frac{1}{2})}$ and $T_p^{(1)}$ may be dealt with separately. Considering first $T^{(1,-\frac{1}{2})}$, from (3.31) the corresponding particular solution to (2.11, 2.12) is obtained,

$$v^{(1,-\frac{1}{2})} = \frac{\sigma\sqrt{2}}{\pi^2}\sum_{n=0}^{\infty}\frac{(-)^n\sinh(2n+1)\pi x\cos(2n+1)\pi z}{(2n+1)^2\cosh(2n+1)\frac{1}{2}\pi}. \tag{3.32}$$

Since $\cos(\pm(2n+1)\frac{1}{2}\pi) = 0$, $v^{(1,-\frac{1}{2})}$ satisfies boundary conditions at $z = \pm\frac{1}{2}$, no additional boundary layer contributions being necessary. However, boundary layers are necessary at the side walls, since $v^{(1,-\frac{1}{2})}$ given by (3.32) does not satisfy the boundary conditions at $x = \pm\frac{1}{2}$. A boundary-layer contribution to the zonal velocity implies a related contribution to the stream function. By (3.11), ${}_{(1)}\psi^{(1,-\frac{1}{6})}$ is found to be required by ${}_{(1)}v^{(1,-\frac{1}{2})}$. These fields together must satisfy (3.7, 3.8). The appropriate boundary conditions at $\xi = 0$ are

$${}_{(1)}\psi^{(1,-\frac{1}{6})} = 0, \quad {}_{(1)}v^{(1,-\frac{1}{2})} = -\frac{\sigma\sqrt{2}}{\pi^2}\sum_{n=0}^{\infty}a_n\cos(2n+1)\pi z,$$

where, by (3.32),

$$a_n = (-)^n\sinh(2n+1)(-\tfrac{1}{2}\pi)\cosh(2n+1)\tfrac{1}{2}\pi\,(2n+1)^{-2}. \tag{3.33}$$

The solutions are

$$\begin{aligned}{}_{(1)}v^{(1,-\frac{1}{2})} = \left(\frac{2\sigma}{3\sqrt{3}}\right)\sum_{n=0}^{\infty}a_n\Big\{&\sqrt{3}\exp\{-[(2n+1)\pi]^{\frac{1}{3}}\xi\}+\tfrac{1}{2}\exp\{-[(2n+1)\pi]^{\frac{1}{3}}\}\frac{\xi}{2}\\ &\times\left(\sqrt{3}\cos[(2n+1)\pi]^{\frac{1}{3}}\frac{\sqrt{3}}{2}\xi+\sin[(2n+1)\pi]^{\frac{1}{3}}\frac{\sqrt{3}}{2}\xi\right)\Big\}\cos(2n+1)\pi z,\end{aligned} \tag{3.34}$$

$$\begin{aligned}{}_{(1)}\psi^{(1,-\frac{1}{6})} = \left(\frac{2\sigma}{3\sqrt{3}\,\pi^{\frac{2}{3}}}\right)\sum_{n=0}^{\infty}a_n(2n+1)^{-\frac{2}{3}}\Big\{&\sqrt{3}\exp\{-[(2n+1)\pi]^{\frac{1}{3}}\xi\}\\ &+\tfrac{1}{2}\exp\{-[(2n+1)\pi]^{\frac{1}{3}}\}\frac{\xi}{2}\left(-\sqrt{3}\cos[(2n+1)\pi]^{\frac{1}{3}}\frac{\sqrt{3}}{2}\xi\right.\\ &\left.+\sin[(2n+1)\pi]^{\frac{1}{3}}\frac{\sqrt{3}}{2}\xi\right)\Big\}\sin(2n+1)\pi z.\end{aligned} \tag{3.35}$$

The existence of these boundary-layer solutions indicates that the zonal velocity as given by (3.32) is the correct total $v^{(1)}$ for the interior of the fluid. Isolines of $v^{(1,-\frac{1}{2})}$ are seen in §3.4 to be essentially horizontal. Thus the zonal velocity v, to first order, will be tilted away from the purely horizontal zero-order thermal wind by $v^{(1,-\frac{1}{2})}$, much as the first-order temperature is tilted from the vertical by $T^{(1,-\frac{1}{2})}$.

The stream function ${}_{(1)}\psi^{(1,-\frac{1}{6})}$ given by (3.35) is associated with two narrow cells of velocity confined to each side-wall boundary-layer region, one extending from $z = -\frac{1}{2}$ to $z = 0$, and the other from $z = 0$ to $z = +\frac{1}{2}$. $\psi^{(1,-\frac{1}{2})}$ exists only in region 1 and is consistently associated with no net mass transport. The additional contributions to the first-order velocities forced by $T_p^{(1)}$ can be seen also to exist only in region 1, and therefore not to affect net transports. In region 1, $T_{px}^{(1)} = O(\epsilon^{-\frac{1}{3}})$ and will only force amplitudes of $\Psi^{(1)}$ and $v^{(1)}$ smaller than those already calculated. However, these velocities are at best of interest only *per se*; and the involved calculations will not be performed. The results given by (3.34, 3.35) were obtained because they were necessary to establish the results for the first-order velocity in the interior.

3.3. *Range of validity of the theory*

It is of course impossible to prove and to define the range of the convergence of the double perturbation expansions for the solutions with mathematical rigour. It is, however, useful and necessary to place plausible restrictions on the applicability of the theory. This will be done in terms of the magnitude of the largest neglected terms. Therefore, although no detailed calculations of the complicated second-order fields will be made, an examination of the magnitude of the second-order temperature is given.

The equation for the second-order temperature is found to be

$$-\epsilon\nabla^2 T^{(2)} + \sigma(\Psi_z^{(0)}T_x^{(1)} - \Psi_x^{(0)}T_z^{(1)} + \Psi_z^{(1)}T_x^{(0)} - \Psi_x^{(1)}T_z^{(0)}) = 0. \tag{3.36}$$

The last term will be zero everywhere, since $T_z^{(0)} = 0$. The remaining convection terms are also zero in the interior, since the interior stream function, though first order, remains constant; conduction alone still determines the interior temperature through second-order terms. In each boundary layer, expanding the convection terms in ϵ yields many contributions to $T^{(2)}$ differing only by terms $O(\epsilon^{\frac{1}{3}})$. Note that convection terms involving the vertical gradient of the temperature, which were identically zero in the first-order temperature equation, appear in (3.36). However, all the amplitudes of $T^{(2)}$ required for the particular solution to (3.36) are found to be of higher order in ϵ than the homogeneous solution required to satisfy the insulating condition along the top and bottom walls. This is precisely similar to the first-order result.

The amplitude of $T^{(2)}$ thus required is $O(\epsilon^{-1})$, N.B. $\beta^2\epsilon^{-1} = (\beta\epsilon^{-\frac{1}{2}})^2$. The ϵ-independent amplitude of $T^{(2,-1)}$ is determined by the condition on the gradient at $z = -\frac{1}{2}$, which is

$$\frac{\partial}{\partial z}T^{(2,-1)}|_{z=-\frac{1}{2}} = -\sigma T_x^{(1,-\frac{1}{3})}(x, -\tfrac{1}{2})\frac{\partial}{\partial \zeta}v^{(0,0)}|_{\zeta=0}. \tag{3.37}$$

The right-hand side of (3.37) is a known function of x. In order to obtain a numerical estimate of the amplitude of $T^{(2,-1)}$, the absolute value of (3.37) is averaged over x, yielding

$$\left(\left|\frac{\partial}{\partial z}T^{(2,-1)}|_{z=-\frac{1}{2}}\right|\right)_{\text{avg}} = \left(\frac{\sigma}{2\sqrt{2}}\right)^2 O(1). \tag{3.38}$$

The order unity quantity in (3.38) is found from (3.31), or from figure 6, to be approximately $\frac{3}{4}$.

This result indicates that the critical parameter for the double expansion procedure is actually

$$\lambda = (2\sqrt{2})^{-1}\sigma\beta\epsilon^{-\frac{1}{2}} = (2\sqrt{2})^{-1}\alpha g(\Delta T)\nu^{\frac{1}{2}}\kappa^{-1}(2\Omega)^{-\frac{3}{2}}. \tag{3.39}$$

Thus convergence is indicated for $\lambda < 1$. To make second-order terms less than 10 % of first-order terms, $\lambda^2 = (0{\cdot}36)^2(\sigma\beta\epsilon^{-\frac{1}{2}})^2 < 0{\cdot}1$, i.e. $\sigma\beta\epsilon^{-\frac{1}{2}}$ may be only slightly less than 1. A safe restriction to indicate that the first-order fields are correct to 10 % is $\sigma\beta\epsilon^{-\frac{1}{2}} < \frac{3}{4}$.

The original restriction $\beta < 1$, $\epsilon < 1$ must of course still be retained. Now that boundary-layer widths are known in terms of ϵ, further restrictions may be placed on this parameter. Requiring that the broad side-wall boundary layers be less than 15 % of the total non-dimensional distance, we must have $\epsilon^{\frac{1}{4}} < 0{\cdot}15$, i.e. $\epsilon < 3\times 10^{-3}$.

In summary, the limitations are

$$\beta < 1, \quad \epsilon < 3\times 10^{-3}, \quad \sigma\beta\epsilon^{-\frac{1}{2}} < \tfrac{3}{4}. \tag{3.40}$$

For a given fluid (ν), in a given geometrical situation (L), a minimum bound on ϵ (maximum bound on Ω), may be obtained to maintain the validity of the original assumption, (f), of negligible centrifugal effects. The range of parameters allowed by these considerations indicates that the first-order theory should be valid for the lower symmetric régime in most situations attainable in the laboratory (see Fultz 1956, p. B-84).

3.4. *Results*

The lower symmetric régime of a horizontally heated rotating fluid, confined to a square cross-section annulus whose dimensions are small compared to the distance from the centre of rotation, has been discussed. The solutions for the non-linearly coupled temperature and velocity fields have been obtained by first expanding in the thermal Rossby number, $\beta = \alpha g(\Delta T)(2\Omega)^{-2}L^{-1}$, and then, to each order in β, performing boundary-layer analysis with respect to the reciprocal of the Taylor number squared, $\epsilon = \nu(2\Omega)^{-1}L^{-2}$.

In the range of parameters appropriate to this procedure, it has been seen that the non-linear convective terms in the heat equation significantly control the development of the fields, while the non-linear momentum advections (the inertial terms in the Navier–Stokes equations) are negligible. Because of the perturbation analyses the parameters β and ϵ do not appear in the approximate equations, which contain only the Prandtl number, $\sigma = \nu\kappa^{-1}$. Since inertial terms are negligible, σ appears only as a factor in the amplitude of all fields of higher β-order than zero. The mathematical problem solved is the same as that which would have arisen had the assumptions been made that $\beta = 0$, $(\beta\sigma) \neq 0$, with an accompanying expansion in $\beta\sigma$. This would be a valid approach for $\sigma > 1$, as it is for laboratory experiments with water. However, the approach actually used is more revealing in that the effects of the inertial terms are examined and the modifications occurring from their inclusion can be estimated in terms of ϵ and σ.

The theory will be most useful when the conditions (3.40) are satisfied. We then have that the temperature and zonal velocity are described by terms up to first order in β. The boundary-layer velocities are adequately described by the zero-order terms in β, first-order effects being entirely confined to internal modifications in the side-wall boundary regions. The meridional and vertical velocity components are identically zero in the interior up to the first order.

The dependence on β, ϵ and σ has been extracted so that these parameters determine only amplitudes and scaled co-ordinates. Thus, except for scaling, the results of any particular experiment are fully described in terms of functions devoid of physical dimensions. Figures 3–8 are graphs of the numerical functions most useful for comparison with experiment.

The zero-order side-wall boundary-layer vertical and meridional velocity streamlines, from the expressions (3.25, 3.27), are shown in figures 3 and 4. The streamline distortion associated with ${}_{(1)}\psi^{(0,\frac{1}{2})}$ is a result of Coriolis effects as well as the appearance of the additional cell implied by ${}_{(1)}\psi^{(0,\frac{1}{3})}$. The total zero-order streamlines are a linear superposition of these two contributions,

$$ {}_{(1)}\psi^{(0)} = \epsilon^{\frac{1}{3}}{}_{(1)}\psi^{(0,\frac{1}{3})} + \epsilon^{\frac{1}{2}}{}_{(1)}\psi^{(0,\frac{1}{2})}. $$

The cell velocity strengthens, the mass-transporting velocity near the wall and weakens it away from the wall. The extra factor of $\epsilon^{-\frac{1}{6}}$ tends to make ${}_{(1)}\psi^{(0,\frac{1}{3})}$ dominant, but from figures 3 and 4 the purely numerical amplitude of ${}_{(1)}\psi^{(0,\frac{1}{3})}$

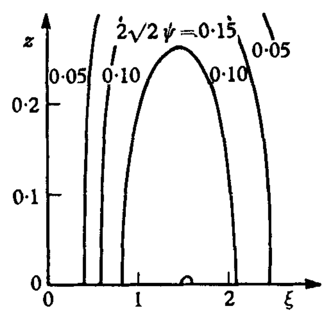

FIGURE 3. Streamlines of ${}_{(1)}\psi^{(0,\frac{1}{3})}$.

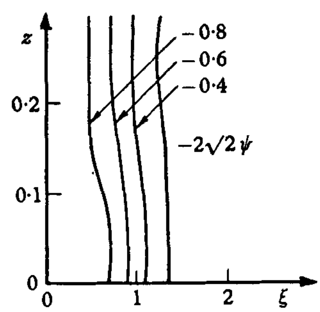

FIGURE 4. Streamlines of ${}_{(1)}\psi^{(0,\frac{1}{2})}$.

is seen to be less than that of ${}_{(1)}\psi^{(0,\frac{1}{2})}$. Thus for low rotation rates (large ϵ), the counter current associated with ${}_{(1)}\psi^{(0,\frac{1}{3})}$ may or may not be observable, but it definitely will be observable for large rotation rates. The boundary-layer streamlines along the top and bottom walls are straight lines given by (3.17) and (3.23).

The zonal velocity would of course vanish if there were no rotation. The relation of the zonal velocity to Coriolis effects is easily seen in the zero-order results. In the top and bottom boundary layers where the cross-rotation-lines meridional velocity is the largest, the velocity-dependent Coriolis accelerations are the largest and the zonal velocity attains its maximum and minimum values. In the interior, the pressure balances the Coriolis accelerations due to the zonal velocity together with the thermally induced gravitational body force. In order for the interior pressure to do this consistently, the vertical zonal velocity gradient must equal the horizontal temperature gradient; this is the thermal wind relationship. Consistent with the zero-order linear horizontal conduction temperature is a zero-order linear vertical zonal velocity distribution. The contributions to the zero-order zonal velocity in the side-wall boundary layers, ${}_{(1)}v^{(0)} = {}_{(1)}v^{(0,0)} + \epsilon^{\frac{1}{8}}\,{}_{(1)}v^{(0,\frac{1}{8})}$, are given by (3.26, 3.28) and are shown in figures 5 and 6.

The observable first-order temperature has been found to be directly set by conduction, although it satisfies boundary conditions determined by the first effects of convection. This temperature, $T^{(1,-\frac{1}{2})}$, is given by (3.31) and shown in figure 7. $T^{(1,-\frac{1}{2})}$ predominantly governs the vertical temperature structure. Therefore the actual vertical temperature gradient of interest to the stability problem is seen to be proportional to $(\alpha g \nu^{\frac{1}{2}} \kappa^{-1})\, L^{-1} (\Delta T)^2\, \Omega^{-\frac{3}{2}}$ degrees Centigrade

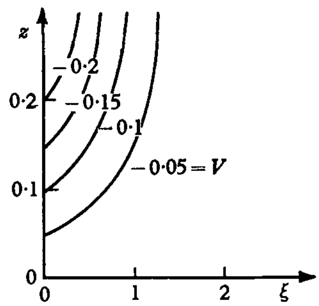

FIGURE 5. Isolines of ${}_{(1)}v^{(0,0)}$.

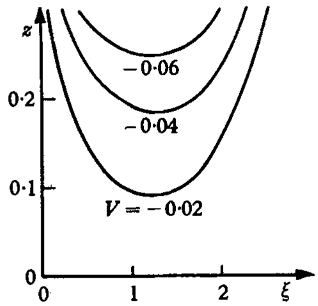

FIGURE 6. Isolines of ${}_{(1)}v^{(0,\frac{1}{8})}$.

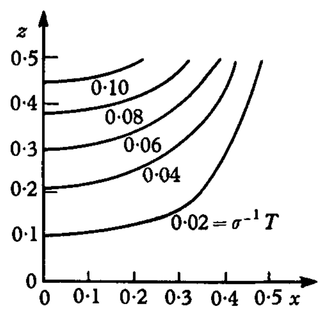

FIGURE 7. Isotherms of $\sigma^{-1} T^{(1,-\frac{1}{2})}$.

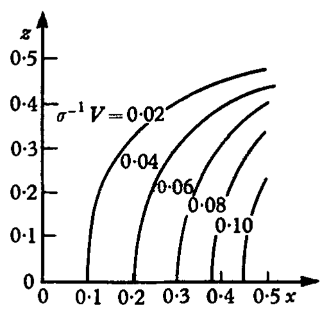

FIGURE 8. Isolines of $\sigma^{-1} v^{(1,-\frac{1}{2})}$.

per centimetre; this is a second-order effect in the temperature difference, and varies with the inverse three-halves power of the rotation rate. Figure 8 shows the first-order zonal velocity consistent with $T^{(1,-\frac{1}{2})}$ given by (3.32).

The macroscopic quantity of greatest physical interest which characterizes the system is the total heat transport. A measure of this quantity is the integrated normal temperature gradient along either of the constant temperature side walls. Internally, the fluid transports the heat injected at the hot wall both by conduction and by convection in the boundary-layered velocity cell. *A priori*, one would suppose that the fluid system would be a more efficient total heat transporter than a solid system of identical geometry and conductivity. However, to first order this is not so. Because of the vertical asymmetry of the first-order temperature [equations (3.26, 3.28, 3.31)], $\int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{\partial}{\partial x} T^{(1)}(\pm\tfrac{1}{2}, z)\, dz = 0$, and the macroscopic heat transfer does not differ from the purely conductive zero-order transport. However, the first-order effects redistribute the gradient along the walls so that the effective heat source is lowered at the hot wall and the effective sink is raised at the cold wall. The second-order temperature will not possess the vertical asymmetry of the first order and will modify the heat transport, but for the range of parameters considered above, this will be a very small effect.

3.5. *Concluding remarks*

The relation to experiment. Unfortunately, at the present time there exist no experiments with which the theory can be quantitatively compared. However, such experiments are planned for the near future in the rotating tank laboratory at the Woods Hole Oceanographic Institution under the direction of Dr Alan Faller. An apparatus has been designed for experiments with horizontal heating of a rotating fluid. The design is such that precise and unambiguous descriptions of the various régimes and their instabilities can be obtained, and also such that comparison with the theory initiated here will be possible. An annulus of 8 cm square cross-section is mounted on a rotating platform approximately 1 m from the centre of rotation. The side walls of the annulus are copper, the bottom is plywood and the top is plexiglass. Horizontal temperature contrasts from $\frac{1}{10}$ °C upward can be maintained by constant-temperature baths; the rotation rate is continuously variable up to well over 1 rad/s. The working fluid is water.

In pilot experiments with this apparatus the lower symmetric régime has been found to exist for values of $\Delta T < 0{\cdot}2 \sim 0{\cdot}3$ °C. Thus a sample point conveniently attainable experimentally and well within the convergence of the theory is

$$\Delta T = 0{\cdot}10\,^{\circ}\mathrm{C}, \quad \Omega = 1{\cdot}0\,\mathrm{sec}^{-1}.$$

For water at 25 °C,

$$\alpha = 2{\cdot}62\times 10^{-4}\,^{\circ}\mathrm{C}^{-1}, \quad v = 8{\cdot}96\times 10^{-3}\,\mathrm{cm}^2.\mathrm{sec}^{-1} \quad \text{and} \quad \sigma = 6{\cdot}28.$$

For $L = 8$ cm. and $g = 980\,\mathrm{cm.sec}^{-2}$, $\beta = 8{\cdot}0\times 10^{-4}$, $\epsilon = 0{\cdot}70\times 10^{-4}$, and $\beta\sigma\epsilon^{-\frac{1}{2}} = 0{\cdot}60$. In figures 9 and 10 the temperature distribution over the body of the fluid and the streamlines in the side-wall boundary layer are shown at these values of the parameters.

The boundary conditions. It is interesting to note the relative role played by the boundary conditions along the vertical and the horizontal surfaces in controlling the behaviour of the system. The imposed horizontal gradient does drive the system, but otherwise the details of the evolved fields are strongly controlled by the conditions along the top and bottom walls. More precisely, after the zero-order temperature calculation, a consideration of the side-wall boundary region was the last step in the calculation of any field, and was done consistently with the requirements from the interior and horizontal boundary-layer region. The fact that (in this range of Rossby and Taylor numbers) the details of the side-wall boundary layers do not affect the fields in the other regions is of vital interest to the philosophical question of the validity of modelling the atmosphere in a laboratory tank. The side walls, which cannot be eliminated or made non-rigid, are the most artificial aspect of the modelling.

On the other hand, the influence of the horizontal wall boundary conditions on the velocity and temperature throughout the fluid is strong. The counter-current along the side walls has been traced to the rotational constraint manifest in the lack of freedom in the solutions of the horizontal boundary-layer equations after application of boundary conditions at the top and bottom walls. There may be other situations of geophysical interest where a counter current along one boundary, not at all associated with the mass-transporting flow, is intimately connected with the detailed conditions along another boundary.

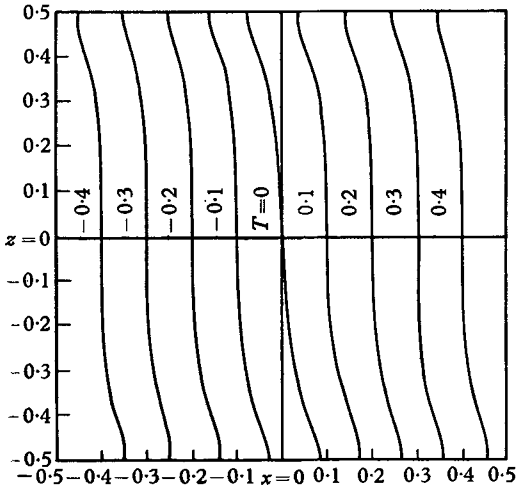

FIGURE 9. Isotherms, $\epsilon = 0{\cdot}7 \times 10^{-4}$, $\beta = 8 \times 10^{-4}$, $\sigma = 6{\cdot}3$.

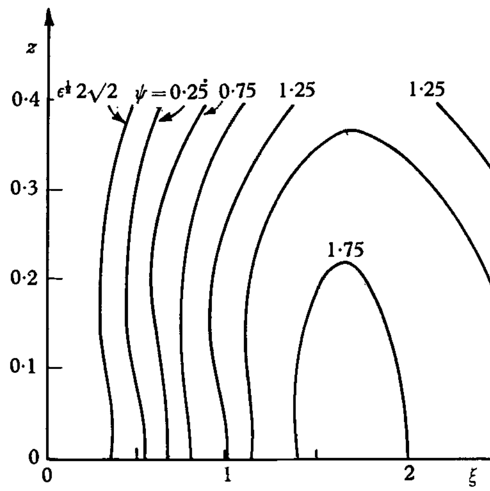

FIGURE 10. Side-wall boundary-layer streamlines, $\epsilon = 7 \times 10^{-4}$, $\beta = 8 \times 10^{-4}$, $\sigma = 6{\cdot}3$.

The temperature distribution to first and higher orders has been seen to be dominated by the insulating condition on the horizontal walls. A change of temperature boundary conditions along the top and bottom walls would therefore markedly alter all results, e.g. if the condition were placed on the temperature itself instead of on the temperature gradient, the amplitude of the first-order temperature and therefore of the interior vertical gradient would be $O(\beta)$, not $O(\beta\epsilon^{-\frac{1}{2}})$, as has been found. A detailed consideration of a change of boundary conditions (on the vertical walls as well) would be of interest. The resulting mathematical problems could be readily treated by the technique developed here. The problem resulting from changing the vertical surfaces from perfectly insulating to perfectly conducting is essentially included in the above work. The temperature along the top and bottom walls would then be a linear function of the horizontal co-ordinate between the hot and cold wall temperatures. The first-order temperature would be entirely given by the particular solution, $T_p^{(1)}$, of § 2, which was entirely obscured in the homogeneous solution, $T^{(1,-\frac{1}{2})}$.

It is a pleasure to thank Professor George F. Carrier for his very valuable advice and criticism. I should also like to thank both Dr Willem V. R. Malkus, who suggested the problem, and Dr George Veronis for many interesting discussions. The material presented here is part of a doctoral thesis presented to the Department of Physics, Harvard University. I should like to thank the Woods Hole Oceanographic Institution for fellowships held during the time of this research.

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CORRIGENDA

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(*J. Fluid Mech.* **5**, 1959, 401.)

In the discussion following equation (2.21) it is incorrectly stated that overstability cannot occur when the Prandtl number σ exceeds the value $\sqrt{\frac{2}{3}}$. As Prof. Chandrasekhar has pointed out to me, the Rayleigh number in the overstable case has no minimum for $\sigma > \sqrt{\frac{2}{3}}$, but it is certainly possible for equation (2.20) to be satisfied for this range of σ. As long as $\sigma < 1$ there always exist values of α^2 for which equation (2.20) can be satisfied.

I neglected to include in the bibliography the reference to:

CHANDRASEKHAR, S. & ELBERT, D. D. 1955 The instability of a layer of fluid heated below and subject to Coriolis forces. II. *Proc. Roy. Soc.* A, **231**, 198.

This investigation treated the stability problem with rigid boundaries for the overstable motions. Figure 5 of my paper is the same as figure 1 in Chandrasekhar & Elbert for the range $\sigma < 0{\cdot}677$.

G. VERONIS

'Energy content and ionization level in an argon gas jet heated by a high intensity arc.' (*J. Fluid Mech.* **4**, 1958, 529.)

The left-hand side of equation (4) should be replaced by its logarithm to base e.

G. L. CANN and A. C. DUCATI